

CLASSIFICATION OF RESOLVING SUBCATEGORIES AND GRADE CONSISTENT FUNCTIONS

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ABSTRACT. We classify certain resolving subcategories of finitely generated modules over a commutative noetherian ring R by using integer-valued functions on $\operatorname{Spec} R$. As an application we give a complete classification of resolving subcategories when R is a locally hypersurface ring. Our results also recover a “missing theorem” by Auslander.

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1. INTRODUCTION

Let R be a commutative noetherian ring with identity which is not necessarily of finite Krull dimension. All modules are assumed to be finitely generated. We denote by $\mathbf{mod} R$ the category of (finitely generated) R -modules. A resolving subcategory of $\mathbf{mod} R$ is a full subcategory that contains projective modules and is closed under direct summands, extensions and syzygies. In this paper we give a classification of certain resolving subcategories of $\mathbf{mod} R$ using functions from $\operatorname{Spec} R$ to the integers. Our results can be applied to obtain a complete classification of all resolving subcategories when R is a locally hypersurface ring.

There has been a flurry of research activities on classification of subcategories associated to algebraic objects in recent years. These results are becoming increasingly influential in ring theory, homotopy theory, algebraic geometry and representation theory. For instance, Gabriel [12] classified all Serre subcategories of $\mathbf{mod} R$ using specialization closed subsets of $\operatorname{Spec} R$. Devinatz, Hopkins and Smith [11, 16] classified thick subcategories in the

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stable homotopy category. Hopkins [15] and Neeman [18] gave a similar result for thick subcategories of the derived categories of perfect complexes over R . The Hopkins-Neeman Theorem was extended to schemes by Thomason [25] and very recently to group algebras by Benson, Iyengar and Krause [6]. For many other similar results, see [5, 17, 19, 21, 23] and the references therein.

Generally speaking, the less restrictive conditions one imposes on the subcategories the harder it is to classify them. A difficulty is the fact that $\mathbf{mod} R$ is “too big”, even when R is nice (say, regular local). Thus attentions are naturally restricted to more special categories such as that of maximal Cohen-Macaulay modules, or derived categories. However, it has recently emerged that resolving categories of $\mathbf{mod} R$ are reasonable objects to look at, and they capture substantial information on the singularities of R , see [9, 10]. Therefore, it is natural to ask if one can classify all such subcategories.

Our main results indicate that such task is quite subtle but at the same time not hopeless. The key new insight is to use certain integer-valued functions on $\mathbf{Spec} R$ and not just the subsets of it. Let us describe such a class of functions which will be crucial for the rest of the paper. Let \mathbb{N} denote the set of nonnegative integers. Let f be an \mathbb{N} -valued function on $\mathbf{Spec} R$. We call f *grade consistent* if it satisfies the following two conditions.

- For all $\mathfrak{p} \in \mathbf{Spec} R$ one has $f(\mathfrak{p}) \leq \text{grade } \mathfrak{p}$.
- For all $\mathfrak{p}, \mathfrak{q} \in \mathbf{Spec} R$ with $\mathfrak{p} \subseteq \mathfrak{q}$ one has $f(\mathfrak{p}) \leq f(\mathfrak{q})$.

We denote by $\text{PD}(R)$ the subcategory of $\mathbf{mod} R$ consisting of modules of finite projective dimension. One of our main results states

Theorem 1.1. *Let R be a commutative noetherian ring. There exists a one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{Resolving subcategories of } \mathbf{mod} R \\ \text{contained in } \text{PD}(R) \end{array} \right\} \begin{array}{c} \xrightarrow{\phi} \\ \xleftarrow{\psi} \end{array} \left\{ \begin{array}{c} \text{Grade consistent functions} \\ \text{on } \mathbf{Spec} R \end{array} \right\}.$$

Here ϕ, ψ are defined as follows:

$$\begin{aligned} \phi(\mathcal{X}) &= [\mathfrak{p} \mapsto \max_{X \in \mathcal{X}} \{\text{pd } X_{\mathfrak{p}}\}], \\ \psi(f) &= \{M \in \mathbf{mod} R \mid \text{pd } M_{\mathfrak{p}} \leq f(\mathfrak{p}) \text{ for all } \mathfrak{p} \in \mathbf{Spec} R\}. \end{aligned}$$

For $M \in \mathbf{mod} R$ we denote by $\mathbf{add} M$ the subcategory of $\mathbf{mod} R$ consisting of modules isomorphic to direct summands of finite direct sums of copies of M . We say that a resolving subcategory \mathcal{X} of $\mathbf{mod} R$ is *dominant* if for every $\mathfrak{p} \in \mathbf{Spec} R$ there exists $n \geq 0$ such that $\Omega^n \kappa(\mathfrak{p}) \in \mathbf{add} \mathcal{X}_{\mathfrak{p}}$, where $\mathcal{X}_{\mathfrak{p}}$ denotes the subcategory of $\mathbf{mod} R_{\mathfrak{p}}$ consisting of modules of the form $X_{\mathfrak{p}}$ with $X \in \mathcal{X}$. Over a Cohen-Macaulay ring, the grade consistent functions also classify the dominant resolving subcategories.

Theorem 1.2. *Let R be a Cohen-Macaulay ring. Then one has the following one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{Dominant resolving subcategories} \\ \text{of } \mathbf{mod} R \end{array} \right\} \begin{array}{c} \xrightarrow{\phi} \\ \xleftarrow{\psi} \end{array} \left\{ \begin{array}{c} \text{Grade consistent functions} \\ \text{on } \mathbf{Spec} R \end{array} \right\},$$

where ϕ, ψ are defined by:

$$\begin{aligned}\phi(\mathcal{X}) &= [\mathfrak{p} \mapsto \text{ht } \mathfrak{p} - \min_{X \in \mathcal{X}} \{\text{depth } X_{\mathfrak{p}}\}], \\ \psi(f) &= \{ M \in \text{mod } R \mid \text{depth } M_{\mathfrak{p}} \geq \text{ht } \mathfrak{p} - f(\mathfrak{p}) \text{ for all } \mathfrak{p} \in \text{Spec } R \}.\end{aligned}$$

For an R -module M we denote by $\text{IPD}(M)$ the *infinite projective dimension locus* of M , i.e., the set of prime ideals \mathfrak{p} with $\text{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \infty$. For a subcategory \mathcal{X} of $\text{mod } R$, set $\text{IPD}(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} \text{IPD}(X)$. Let $\text{Sing } R$ denote the *singular locus* of R , that is, the set of prime ideals \mathfrak{p} of R such that $R_{\mathfrak{p}}$ is not a regular local ring. Let W be a subset of $\text{Sing } R$. We say that W is *specialization closed* if $V(\mathfrak{p}) \subseteq W$ for every $\mathfrak{p} \in W$. We denote by $\text{IPD}^{-1}(W)$ the subcategory of $\text{mod } R$ consisting of modules whose infinite projective dimension loci are contained in W . Recall that R is said to be *locally hypersurface* if for each $\mathfrak{p} \in \text{Spec } R$ the local ring $R_{\mathfrak{p}}$ is a hypersurface. The above results allow us to give complete classification of resolving subcategories of $\text{mod } R$ when R is a locally hypersurface ring:

Theorem 1.3. *Let R be a locally hypersurface ring. There exists a one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{Resolving subcategories} \\ \text{of } \text{mod } R \end{array} \right\} \begin{array}{c} \xrightarrow{\Phi} \\ \xleftarrow{\Psi} \end{array} \left\{ \begin{array}{c} \text{Specialization closed} \\ \text{subsets of } \text{Sing } R \end{array} \right\} \times \left\{ \begin{array}{c} \text{Grade consistent} \\ \text{functions on } \text{Spec } R \end{array} \right\}.$$

Here Φ, Ψ are defined as follows:

$$\begin{aligned}\Phi(\mathcal{X}) &= (\text{IPD}(\mathcal{X}), \phi(\mathcal{X})), \text{ where } \phi(\mathcal{X}) = [\mathfrak{p} \mapsto \text{ht } \mathfrak{p} - \min_{X \in \mathcal{X}} \{\text{depth } X_{\mathfrak{p}}\}], \\ \Psi(W, f) &= \{ M \in \text{IPD}^{-1}(W) \mid \text{depth } M_{\mathfrak{p}} \geq \text{ht } \mathfrak{p} - f(\mathfrak{p}) \text{ for all } \mathfrak{p} \in \text{Spec } R \}.\end{aligned}$$

For an R -module M we denote by $\text{res } M$ the *resolving closure* of M , that is, the smallest resolving subcategory of $\text{mod } R$ containing M . As another interesting application we can recover a “missing” result by Auslander. One says that *Tor-rigidity* holds for $\text{PD}(R)$ if for any module $M \in \text{PD}(R)$ and $N \in \text{mod } R$, $\text{Tor}_i^R(M, N) = 0$ forces $\text{Tor}_j^R(M, N) = 0$ for $j > i$. It is known that *Tor-rigidity* holds for $\text{PD}(R)$ when R is regular or a quotient of an unramified regular local ring by a regular element. The equivalence of (1) and (3) in the following theorem for unramified regular local rings was announced by Auslander in his 1962 ICM speech but never published (cf. [1, Theorem 3]):

Theorem 1.4. *Let R be a commutative noetherian ring. Suppose that *Tor-rigidity* holds for $\text{PD}(R_{\mathfrak{p}})$ for all $\mathfrak{p} \in \text{Spec } R$. Consider two modules $M, N \in \text{PD}(R)$. The following are equivalent:*

- (1) $\text{pd } M_{\mathfrak{p}} \leq \text{pd } N_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec } R$.
- (2) $M \in \text{res } N$.
- (3) $\text{Supp Tor}_i^R(M, X) \subseteq \text{Supp Tor}_i^R(N, X)$ for all $i > 0$ and all $X \in \text{mod } R$.

We also note that our methods give concrete information such as generators for the categories considered. For example, see Theorem 2.1, a special case and a crucial step in the proof of Theorem 1.1. Here we are able to describe subcategories of $\text{PD}(R)$ as the smallest extension-closed subcategories containing the transposes of certain syzygies of the residue field.

The paper is organized as follows. Section 2 is devoted to proving Theorem 2.1. Sections 3 and 4 consist of various technical results concerning resolving subcategories of $\text{PD}(R)$ and dominant resolving subcategories respectively. We prove Theorem 1.4 in Section 3. Section 5 gives the proofs of Theorems 1.1 and 1.2. In the final section, Section 6, the previous results are applied to prove Theorem 1.3.

Convention. For an R -module M , we denote its n -th syzygy by $\Omega^n M$ and its transpose by $\text{Tr}M$ (see [2] for the definition of the transpose). Whenever R is local, we define them by using a *minimal* free resolution of M , so that they are uniquely determined up to isomorphism. The depth of the zero R -module is ∞ as a convention.

2. RESOLVING SUBCATEGORIES IN $\text{PD}_0(R)$

Throughout this section, let (R, \mathfrak{m}, k) be a local ring. An *extension-closed* subcategory of $\text{mod } R$ is defined as a subcategory of $\text{mod } R$ which is closed under direct summands and extensions. For an R -module M we denote by $\text{ext } M$ the *extension closure* of M , that is, the smallest extension-closed subcategory of $\text{mod } R$ containing M . We denote by $\text{PD}_0(R)$ the subcategory of $\text{mod } R$ consisting of modules that have finite projective dimension and that are locally free on the *punctured spectrum* $\text{Spec } R \setminus \{\mathfrak{m}\}$. For an integer $n \geq 0$, we denote by $\text{PD}_0^n(R)$ the subcategory of $\text{mod } R$ consisting of modules which are of projective dimension at most n and locally free on the punctured spectrum of R .

This section is devoted to proving the following Theorem which is an important step in proving Theorem 1.1.

Theorem 2.1. *Let R be a commutative noetherian local ring of depth t and with residue field k . Then one has a filtration*

$$\text{add } R = \text{PD}_0^0(R) \subsetneq \text{PD}_0^1(R) \subsetneq \cdots \subsetneq \text{PD}_0^t(R) = \text{PD}_0(R)$$

of resolving subcategories of $\text{mod } R$, and these are all the resolving subcategories contained in $\text{PD}_0(R)$. Moreover, when $t > 0$, for each integer $1 \leq n \leq t$ one has

$$\text{PD}_0^n(R) = \text{res}(\text{Tr}\Omega^{n-1}k) = \text{ext}(R \oplus \bigoplus_{i=0}^{n-1} \text{Tr}\Omega^i k).$$

The key is detailed inspection of syzygies and transposes. We begin with stating several basic properties of syzygies and transposes; the following proposition will be used basically without reference.

Proposition 2.2. *For an R -module M the following hold.*

- (1) *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence of R -modules. Then one has exact sequences:*

$$\begin{aligned} 0 \rightarrow \Omega N \rightarrow L \oplus R^{\oplus a} \rightarrow M \rightarrow 0, \\ 0 \rightarrow \Omega L \rightarrow \Omega M \oplus R^{\oplus b} \rightarrow \Omega N \rightarrow 0, \\ 0 \rightarrow N^* \rightarrow M^* \rightarrow L^* \rightarrow \text{Tr}N \rightarrow \text{Tr}M \oplus R^{\oplus c} \rightarrow \text{Tr}L \rightarrow 0 \end{aligned}$$

for some $a, b, c \geq 0$.

- (2) *There are isomorphisms $(\text{Tr}M)^* \cong \Omega^2 M$ and $M^* \cong \Omega^2 \text{Tr}M \oplus R^{\oplus n}$ for some $n \geq 0$.*
(3) *The module $\text{Tr}M$ has no nonzero free summand.*

- (4) One has an isomorphism $M \cong \text{TrTr}M \oplus R^{\oplus n}$ for some $n \geq 0$. Hence $\text{TrTr}M$ is isomorphic to a maximal direct summand of M without nonzero free summand.
- (5) There is an exact sequence

$$0 \rightarrow \text{Ext}^1(M, R) \rightarrow \text{Tr}M \xrightarrow{f} (\Omega^2 M)^* \rightarrow \text{Ext}^2(M, R) \rightarrow 0$$

of R -modules with $\text{Im } f \cong \Omega \text{Tr} \Omega M$.

Proof. (1) The first sequence is a consequence of a pullback diagram made by $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ and $0 \rightarrow \Omega N \rightarrow R^{\oplus a} \rightarrow N \rightarrow 0$. The second and third sequences are obtained by the horseshoe and snake lemmas from the original sequence.

(2) These isomorphisms are obtained easily by definition.

(3) This statement follows from (the proof of) [24, Lemma 4.2].

(4) The former assertion is straightforward. The latter is by (3).

(5) Applying [2, Proposition (2.6)(a)] to $\text{Tr}M$ and using the first isomorphism in (2), we get such an exact sequence. It follows from [2, Appendix] that the image of f is isomorphic to $\Omega \text{Tr} \Omega M \oplus R^{\oplus n}$ for some $n \geq 0$. We have surjections $\text{Tr}M \rightarrow \Omega \text{Tr} \Omega M \oplus R^{\oplus n} \rightarrow R^{\oplus n}$, which shows that $\text{Tr}M$ has a free summand isomorphic to $R^{\oplus n}$. By (3) we have $n = 0$. ■

Lemma 2.3. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence of R -modules. Let $n \geq 0$ be an integer such that $\text{Ext}^n(L, R) = 0$. Then there is an exact sequence*

$$0 \rightarrow \text{Tr} \Omega^n N \rightarrow \text{Tr} \Omega^n M \oplus R^{\oplus m} \rightarrow \text{Tr} \Omega^n L \rightarrow 0.$$

Proof. We have an exact sequence $0 \rightarrow \Omega^n L \rightarrow \Omega^n M \oplus R^{\oplus l} \rightarrow \Omega^n N \rightarrow 0$. The following three exact sequences are induced:

$$\begin{aligned} (\Omega^n M \oplus R^{\oplus l})^* &\xrightarrow{\alpha} (\Omega^n L)^* \xrightarrow{\beta} \text{Tr} \Omega^n N \rightarrow \text{Tr} \Omega^n M \oplus R^{\oplus m} \rightarrow \text{Tr} \Omega^n L \rightarrow 0, \\ (\Omega^n M \oplus R^{\oplus l})^* &\xrightarrow{\alpha} (\Omega^n L)^* \xrightarrow{\gamma} \text{Ext}^{n+1}(N, R) \xrightarrow{\delta} \text{Ext}^{n+1}(M, R), \\ 0 = \text{Ext}^n(L, R) &\rightarrow \text{Ext}^{n+1}(N, R) \xrightarrow{\delta} \text{Ext}^{n+1}(M, R). \end{aligned}$$

Hence we observe: δ is injective, γ is zero, α is surjective, and β is zero. ■

Proposition 2.4. *Assume $\text{depth } R = t > 0$. Let M be an R -module which is locally free on the punctured spectrum of R . Then for each integer $0 \leq i < t$, there exists an exact sequence*

$$0 \rightarrow \text{Tr} \Omega^{i+1} \text{Tr} \Omega^{i+1} M \rightarrow \text{Tr} \Omega^i \text{Tr} \Omega^i M \oplus R^{\oplus n_i} \rightarrow \text{Tr} \Omega^i \text{Ext}^{i+1}(M, R) \rightarrow 0.$$

Proof. Applying Proposition 2.2(5) to $\Omega^i M$, we have an exact sequence $0 \rightarrow \text{Ext}^{i+1}(M, R) \rightarrow \text{Tr} \Omega^i M \rightarrow \Omega \text{Tr} \Omega^{i+1} M \rightarrow 0$. The assumption implies that $\text{Ext}^{i+1}(M, R)$ has finite length. Since $i < t$, we have $\text{Ext}^i(\text{Ext}^{i+1}(M, R), R) = 0$. The assertion now follows from Lemma 2.3. ■

Lemma 2.5. *Assume $\text{depth } R = t > 0$. Let $L \neq 0$ be an R -module of finite length, and let $0 \leq n < t$ be an integer. Take a minimal free resolution $\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow L \rightarrow 0$ of L . Then one has an exact sequence*

$$0 \rightarrow F_0^* \xrightarrow{d_1^*} \cdots \xrightarrow{d_{n-1}^*} F_{n-1}^* \xrightarrow{d_n^*} F_n^* \xrightarrow{d_{n+1}^*} F_{n+1}^* \rightarrow \text{Tr} \Omega^n L \rightarrow 0,$$

which gives a minimal free resolution of $\text{Tr}\Omega^n L$. Hence,

$$\Omega^i \text{Tr}\Omega^n L \cong \begin{cases} \text{Tr}\Omega^{n-i} L & (0 \leq i \leq n), \\ F_0^* & (i = n+1), \\ 0 & (i \geq n+2). \end{cases}$$

In particular, $\text{pd}(\text{Tr}\Omega^n L) = n+1$.

Proof. There is an exact sequence $0 \rightarrow \Omega^n L \rightarrow F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0 \rightarrow L \rightarrow 0$. Since $n < t$, we have $\text{Ext}^i(L, R) = 0$ for any $i \leq n$. Hence the sequence

$$0 \rightarrow F_0^* \xrightarrow{d_1^*} \cdots \xrightarrow{d_{n-1}^*} F_{n-1}^* \rightarrow (\Omega^n L)^* \rightarrow 0$$

is exact. Splicing this with $0 \rightarrow (\Omega^n L)^* \rightarrow F_n^* \xrightarrow{d_{n+1}^*} F_{n+1}^* \rightarrow \text{Tr}\Omega^n L \rightarrow 0$, we are done. \blacksquare

Proposition 2.6. *Assume $\text{depth } R = t > 0$. Let $L \neq 0$ be an R -module of finite length. Then $\text{res}(\text{Tr}\Omega^n L) = \text{res}(\text{Tr}\Omega^n k)$ for every $0 \leq n < t$.*

Proof. As $L \neq 0$, there is an exact sequence $0 \rightarrow L' \rightarrow L \rightarrow k \rightarrow 0$. Since L' has finite length and $n < t$, we have $\text{Ext}^n(L', R) = 0$, and get an exact sequence

$$0 \rightarrow \text{Tr}\Omega^n k \rightarrow \text{Tr}\Omega^n L \oplus R^{\oplus m} \rightarrow \text{Tr}\Omega^n L' \rightarrow 0$$

by Lemma 2.3. Induction on the length of L shows that $\text{Tr}\Omega^n L$ belongs to $\text{res}(\text{Tr}\Omega^n k)$, which implies

$$(2.6.1) \quad \text{res}(\text{Tr}\Omega^n L) \subseteq \text{res}(\text{Tr}\Omega^n k).$$

(Note that this inclusion relation is still valid for $L = 0$.) The above exact sequence induces an exact sequence

$$(2.6.2) \quad 0 \rightarrow \Omega \text{Tr}\Omega^n L' \rightarrow \text{Tr}\Omega^n k \oplus R^{\oplus l} \rightarrow \text{Tr}\Omega^n L \oplus R^{\oplus m} \rightarrow 0.$$

Let us prove

$$(2.6.3) \quad \text{Tr}\Omega^n k \in \text{res}(\text{Tr}\Omega^n L)$$

by induction on n . When $n = 0$, the module $\Omega \text{Tr}\Omega^n L'$ is free by Lemma 2.5, and (2.6.3) follows from (2.6.2). When $n \geq 1$, we have:

$$\Omega \text{Tr}\Omega^n L' \stackrel{(1)}{\cong} \text{Tr}\Omega^{n-1} L' \stackrel{(2)}{\in} \text{res}(\text{Tr}\Omega^{n-1} k) \subseteq \text{res}(\text{Tr}\Omega^{n-1} L) \stackrel{(3)}{=} \text{res}(\Omega \text{Tr}\Omega^n L) \stackrel{(4)}{\subseteq} \text{res}(\text{Tr}\Omega^n L).$$

Here (1),(4) follow from Lemma 2.5, (2) from (2.6.1), and (3) from the induction hypothesis. Now (2.6.3) is obtained by (2.6.2). Consequently, we have the inclusion $\text{res}(\text{Tr}\Omega^n L) \supseteq \text{res}(\text{Tr}\Omega^n k)$. \blacksquare

Lemma 2.7. *Let M be an R -module such that $\Omega M \in \text{ext}(R \oplus M)$. Then one has $\text{ext}(R \oplus M) = \text{res } M$.*

Proof. It is clear that $\text{ext}(R \oplus M)$ is contained in $\text{res } M$. Let us consider the subcategory

$$\mathcal{X} = \{ N \in \text{ext}(R \oplus M) \mid \Omega N \in \text{ext}(R \oplus M) \} \subseteq \text{ext}(R \oplus M).$$

By assumption \mathcal{X} contains $R \oplus M$, and we easily see that \mathcal{X} is closed under direct summands. Let $0 \rightarrow S \rightarrow T \rightarrow U \rightarrow 0$ be an exact sequence of R -modules with $S, U \in \mathcal{X}$.

Then $T, \Omega S, \Omega U$ are in $\text{ext}(R \oplus M)$, and there is an exact sequence $0 \rightarrow \Omega S \rightarrow \Omega T \oplus R^{\oplus n} \rightarrow \Omega U \rightarrow 0$. Hence ΩT belongs to $\text{ext}(R \oplus M)$, which implies $T \in \mathcal{X}$. Therefore \mathcal{X} is closed under extensions, and thus $\mathcal{X} = \text{ext}(R \oplus M)$. Now $\text{ext}(R \oplus M)$ is closed under syzygies, which means that it is resolving. \blacksquare

Now we can achieve the main purpose of this section.

Proof of Theorem 2.1. First of all, it is easy to observe that there is a filtration

$$\text{add } R = \text{PD}_0^0(R) \subsetneq \text{PD}_0^1(R) \subsetneq \cdots \subsetneq \text{PD}_0^t(R) = \text{PD}_0(R),$$

and that each $\text{PD}_0^n(R)$ is a resolving subcategory of $\text{mod } R$. When $t = 0$, we have $\text{PD}_0(R) = \text{PD}_0^0(R) = \text{add } R$, which is the smallest resolving subcategory of $\text{mod } R$. So, we may assume $t > 0$.

(1) Let us prove the equalities

$$\text{PD}_0^n(R) = \text{res}(\text{Tr}\Omega^{n-1}k) = \text{ext}(R \oplus \bigoplus_{i=0}^{n-1} \text{Tr}\Omega^i k)$$

for each $1 \leq n \leq t$. Lemma 2.5 implies that $\text{Tr}\Omega^{n-1}k$ belongs to $\text{PD}_0^n(R)$. Hence $\text{PD}_0^n(R)$ contains $\text{res}(\text{Tr}\Omega^{n-1}k)$. Let M be an R -module in $\text{PD}_0^n(R)$. By Proposition 2.4 we have

$$\text{res}(\text{Tr}\Omega^i \text{Tr}\Omega^i M) \subseteq \text{res}(\text{Tr}\Omega^{i+1} \text{Tr}\Omega^{i+1} M \oplus \text{Tr}\Omega^i \text{Ext}^{i+1}(M, R))$$

for $0 \leq i < n$. As $\text{Ext}^{i+1}(M, R)$ has finite length, $\text{Tr}\Omega^i \text{Ext}^{i+1}(M, R)$ is in $\text{res}(\text{Tr}\Omega^i k)$ by Proposition 2.6. Since M has projective dimension at most n , the module $\text{Tr}\Omega^n \text{Tr}\Omega^n M$ is free. Lemma 2.5 implies that $\text{Tr}\Omega^i k \cong \Omega^{n-1-i}(\text{Tr}\Omega^{n-1}k) \in \text{res}(\text{Tr}\Omega^{n-1}k)$ for every $0 \leq i \leq n-1$. Also, $\text{Tr}\Omega^0 \text{Tr}\Omega^0 M$ is isomorphic to M up to free summand. Hence we obtain

$$\begin{aligned} \text{res } M &= \text{res}(\text{Tr}\Omega^0 \text{Tr}\Omega^0 M) \subseteq \text{res}(\text{Tr}\Omega^1 \text{Tr}\Omega^1 M \oplus \text{Tr}\Omega^{n-1}k) \subseteq \text{res}(\text{Tr}\Omega^2 \text{Tr}\Omega^2 M \oplus \text{Tr}\Omega^{n-1}k) \\ &\subseteq \cdots \subseteq \text{res}(\text{Tr}\Omega^n \text{Tr}\Omega^n M \oplus \text{Tr}\Omega^{n-1}k) = \text{res}(\text{Tr}\Omega^{n-1}k). \end{aligned}$$

Therefore $\text{PD}_0^n(R) = \text{res}(\text{Tr}\Omega^{n-1}k)$ holds. The equality $\text{res}(\text{Tr}\Omega^{n-1}k) = \text{ext}(R \oplus \bigoplus_{i=0}^{n-1} \text{Tr}\Omega^i k)$ is a consequence of Lemmas 2.5 and 2.7.

(2) Let us prove that every resolving subcategory contained in $\text{PD}_0(R)$ coincides with one of $\text{PD}_0^0(R), \dots, \text{PD}_0^t(R)$. We start by establishing a claim.

Claim. Let $1 \leq n \leq t$. Let X be an R -module in $\text{PD}_0^n(R) \setminus \text{PD}_0^{n-1}(R)$. Then $\text{Tr}\Omega^i L$ belongs to $\text{res } X$ for all R -modules L of finite length and all integers $0 \leq i \leq n-1$.

Proof of Claim. Fix an R -module L of finite length. Set $E = \text{Ext}^n(X, R)$. Since X has projective dimension n , we see that E is a nonzero module of finite length. Proposition 2.6 yields $\text{Tr}\Omega^{n-1}L \in \text{res}(\text{Tr}\Omega^{n-1}k) = \text{res}(\text{Tr}\Omega^{n-1}E)$. It is easy to see that $E \cong \text{Tr}\Omega^{n-1}X$, whence $\text{Tr}\Omega^{n-1}E \cong \text{Tr}\Omega^{n-1} \text{Tr}\Omega^{n-1}X$. Thus:

$$(2.7.1) \quad \text{Tr}\Omega^{n-1}L \in \text{res}(\text{Tr}\Omega^{n-1} \text{Tr}\Omega^{n-1}X).$$

We show the claim by induction on n . When $n = 1$, it follows from (2.7.1) and Proposition 2.2(4). Let $n \geq 2$. Since $\Omega X \in \text{PD}_0^{n-1}(R) \setminus \text{PD}_0^{n-2}(R)$, the induction hypothesis implies that $\text{Tr}\Omega^j L$ is in $\text{res } \Omega X$ for all $0 \leq j \leq n-2$. Hence:

$$(2.7.2) \quad \text{Tr}\Omega^j L \in \text{res } X \quad (0 \leq j \leq n-2).$$

By Proposition 2.4, there are exact sequences

$$0 \rightarrow \mathrm{Tr}\Omega^{j+1}\mathrm{Tr}\Omega^{j+1}X \rightarrow \mathrm{Tr}\Omega^j\mathrm{Tr}\Omega^jX \oplus R^{\oplus m_j} \rightarrow \mathrm{Tr}\Omega^jE_j \rightarrow 0 \quad (0 \leq j \leq n-2),$$

where $E_j = \mathrm{Ext}^{j+1}(X, R)$. As E_j has finite length, $\mathrm{Tr}\Omega^jE_j$ is in $\mathbf{res} X$ for $0 \leq j \leq n-2$ by (2.7.2). Using the above exact sequences, we inductively observe that:

$$(2.7.3) \quad \mathrm{Tr}\Omega^{n-1}\mathrm{Tr}\Omega^{n-1}X \in \mathbf{res} X.$$

Combining (2.7.1), (2.7.2) and (2.7.3) yields that $\mathrm{Tr}\Omega^iL$ belongs to $\mathbf{res} X$ for any integer $0 \leq i \leq n-1$. \square

Now, let $\mathcal{X} \neq \mathbf{add} R$ be a resolving subcategory of $\mathbf{mod} R$ contained in $\mathrm{PD}_0(R)$. Then there exists a unique integer $1 \leq n \leq t$ such that \mathcal{X} is contained in $\mathrm{PD}_0^n(R)$ but not contained in $\mathrm{PD}_0^{n-1}(R)$. We find an R -module $X \in \mathcal{X}$ that does not belong to $\mathrm{PD}_0^{n-1}(R)$. The above claim says that $\mathrm{Tr}\Omega^{n-1}k$ is in $\mathbf{res} X$, and hence in \mathcal{X} . It follows from (1) that $\mathcal{X} = \mathrm{PD}_0^n(R)$. \blacksquare

We close this section by stating a corollary of Theorem 2.1, which will be used later.

Corollary 2.8. *Let R be a local ring. Let M be an R -module in $\mathrm{PD}_0(R)$. Then one has $\mathbf{res} M = \mathrm{PD}_0^n(R)$, where $n = \mathrm{pd} M$.*

Proof. Theorem 2.1 shows that $\mathbf{res} M = \mathrm{PD}_0^m(R)$ for some integer m . Since M is in $\mathbf{res} M$, we have $n \leq m$. Suppose $n < m$. Then M is in $\mathrm{PD}_0^{m-1}(R)$. As $\mathrm{PD}_0^{m-1}(R)$ is resolving, $\mathbf{res} M$ is contained in $\mathrm{PD}_0^{m-1}(R)$. This is a contradiction, which implies $n = m$. \blacksquare

3. RESOLVING SUBCATEGORIES OF MODULES OF FINITE PROJECTIVE DIMENSION

In this section, we study resolving subcategories whose objects are modules of finite projective dimension. Let us start by recalling two useful results, see [9, Lemma 4.5 and 4.6].

Lemma 3.1. *Let R be a local ring. Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence of R -modules. Then one has the equality*

$$\inf\{\mathrm{depth} L, \mathrm{depth} N\} = \inf\{\mathrm{depth} M, \mathrm{depth} N\}.$$

For an R -module M , denote by $\mathrm{NF}(M)$ the *nonfree locus* of M , that is, the set of prime ideals \mathfrak{p} of R such that the localization $M_{\mathfrak{p}}$ is nonfree as an $R_{\mathfrak{p}}$ -module. It is a well-known fact that $\mathrm{NF}(M)$ is a closed subset of $\mathrm{Spec} R$ in the Zariski topology.

The following result enables us to replace in a resolving subcategory a module with a module whose nonfree locus is irreducible. This is a generalization of [22, Theorem 4.3], and will also be used in the next section.

Proposition 3.2. *Let M be an R -module. For every $\mathfrak{p} \in \mathrm{NF}(M)$ there exists $X \in \mathbf{res} M$ satisfying $\mathrm{NF}(X) = V(\mathfrak{p})$ and $\mathrm{depth} X_{\mathfrak{q}} = \inf\{\mathrm{depth} M_{\mathfrak{q}}, \mathrm{depth} R_{\mathfrak{q}}\}$ for all $\mathfrak{q} \in V(\mathfrak{p})$.*

Next we state two results which are essentially proved in [23].

Lemma 3.3. *Let \mathcal{X} be a resolving subcategory of $\mathbf{mod} R$.*

(1) *For each $\mathfrak{p} \in \mathrm{Spec} R$, the subcategory $\mathbf{add} \mathcal{X}_{\mathfrak{p}}$ of $\mathbf{mod} R_{\mathfrak{p}}$ is resolving.*

- (2) Let Z be a nonempty finite subset of $\text{Spec } R$. Let M be an R -module such that $M_{\mathfrak{p}} \in \text{add } \mathcal{X}_{\mathfrak{p}}$ for all $\mathfrak{p} \in Z$. Then there exist exact sequences

$$\begin{aligned} 0 \rightarrow K \rightarrow X \rightarrow M \rightarrow 0, \\ 0 \rightarrow L \rightarrow M \oplus K \oplus R^{\oplus n} \rightarrow X \rightarrow 0 \end{aligned}$$

of R -modules with $X \in \mathcal{X}$, $\text{NF}(L) \subseteq \text{NF}(M)$ and $\text{NF}(L) \cap Z = \emptyset$.

Proof. Note that the results [23, Lemmas 4.6 and 4.8] hold even if the ring R is not local, as the proofs show. The first assertion now follows, and the second one can be shown along the same lines as in the proof of [23, Proposition 4.7]. \blacksquare

Next we investigate the relationship between a module in a resolving subcategory and its localization. Thanks to the following proposition, we can reduce problems on resolving subcategories to the case where the base ring is local.

Proposition 3.4. *Let \mathcal{X} be a resolving subcategory of $\text{mod } R$. The following are equivalent for an R -module M .*

- (1) $M \in \mathcal{X}$.
- (2) $M_{\mathfrak{p}} \in \text{add } \mathcal{X}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec } R$.
- (3) $M_{\mathfrak{m}} \in \text{add } \mathcal{X}_{\mathfrak{m}}$ for all $\mathfrak{m} \in \text{Max } R$.

Proof. (1) \Rightarrow (3) \Rightarrow (2): Localization shows these implications.

(2) \Rightarrow (1): Assume that M is not in \mathcal{X} . Then

$$\{ \text{NF}(N) \mid N \in \text{mod } R \setminus \mathcal{X} \text{ and } N_{\mathfrak{p}} \in \text{add } \mathcal{X}_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in \text{Spec } R \}$$

is a nonempty family of closed subsets of $\text{Spec } R$. Since $\text{Spec } R$ is noetherian, this set has a minimal element $\text{NF}(L)$; see [14, Chapter I, Exercise 1.7]. As $L \notin \mathcal{X}$, we see that L is not projective. Hence $\min \text{NF}(L)$ is nonempty. Applying Lemma 3.3(2) to $\min \text{NF}(L)$, we get exact sequences

$$\begin{aligned} 0 \rightarrow K \rightarrow X \rightarrow L \rightarrow 0, \\ 0 \rightarrow H \rightarrow L \oplus K \oplus R^{\oplus n} \rightarrow X \rightarrow 0 \end{aligned}$$

of R -modules with $X \in \mathcal{X}$, $\text{NF}(H) \subseteq \text{NF}(L)$ and $\text{NF}(H) \cap \min \text{NF}(L) = \emptyset$. Since $\min \text{NF}(L) \neq \emptyset$, we have $\text{NF}(H) \subsetneq \text{NF}(L)$. Fix a prime ideal \mathfrak{p} of R . Lemma 3.3(1) shows that $\text{add } \mathcal{X}_{\mathfrak{p}}$ is a resolving subcategory of $\text{mod } R_{\mathfrak{p}}$. Localizing the above two exact sequences at \mathfrak{p} , we observe that $H_{\mathfrak{p}}$ is in $\text{add } \mathcal{X}_{\mathfrak{p}}$. The minimality of $\text{NF}(L)$ implies that H belongs to \mathcal{X} . It is seen from the second exact sequence above that L is in \mathcal{X} , which is again a contradiction. Consequently, we have $M \in \mathcal{X}$. \blacksquare

The proposition below, which is a corollary of Theorem 2.1, will be a basis of the proof of the main result of this section.

Proposition 3.5. *Let (R, \mathfrak{m}) be local. Let $M \in \text{PD}_0(R)$ and $N \in \text{PD}(R)$. If $\text{pd } M \leq \text{pd } N$, then one has $M \in \text{res } N$.*

Proof. We may assume that M is nonfree. Hence N is also nonfree, and \mathfrak{m} belongs to $\text{NF}(N)$. Proposition 3.2 shows that there exists an R -module $L \in \text{res } N$ with $\text{NF}(L) = \{\mathfrak{m}\}$

and $\text{depth } L = \inf\{\text{depth } N, \text{depth } R\}$. Since N has finite projective dimension, so is L . We have

$$\begin{aligned} \text{pd } L &= \text{depth } R - \text{depth } L = \text{depth } R - \inf\{\text{depth } N, \text{depth } R\} \\ &= \sup\{\text{pd } N, 0\} = \text{pd } N \geq \text{pd } M. \end{aligned}$$

Thus, replacing N with L , we may assume that N is in $\text{PD}_0(R)$. It follows from Corollary 2.8 that $\text{res } N = \text{PD}_0^n(R)$, where $n = \text{pd } N$. We have $\text{pd } M \leq \text{pd } N = n$, and therefore M belongs to $\text{PD}_0^n(R) = \text{res } N$. \blacksquare

Let $\text{LPD}(R)$ denote the subcategory of $\text{mod } R$ consisting of modules M such that $\text{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty$ for all $\mathfrak{p} \in \text{Spec } R$. The following theorem is the main result of this section, which gives a criterion for a module in $\text{LPD}(R)$ to belong to a resolving subcategory in $\text{LPD}(R)$.

Theorem 3.6. *Let \mathcal{X} be a resolving subcategory of $\text{mod } R$ contained in $\text{LPD}(R)$. Then the following are equivalent for an R -module M .*

- (1) *One has $M \in \mathcal{X}$.*
- (2) *For every $\mathfrak{p} \in \text{NF}(M)$ there exists $X \in \mathcal{X}$ such that $\text{pd } M_{\mathfrak{p}} \leq \text{pd } X_{\mathfrak{p}}$.*

Proof. The implication (1) \Rightarrow (2) is trivial. To prove the opposite implication, assume that the condition (2) holds. By Lemma 3.3(1) and Proposition 3.4, we may assume that (R, \mathfrak{m}) is local. Then we have $\text{LPD}(R) = \text{PD}(R)$. Let us deduce (1) by induction on $m := \dim \text{NF}(M)$.

When $m = -\infty$, the module M is free, and belongs to \mathcal{X} . When $m = 0$, we have $\text{NF}(M) = \{\mathfrak{m}\}$. Hence M is in $\text{PD}_0(R)$, and there exists $X \in \mathcal{X}$ with $\text{pd } M \leq \text{pd } X$. Proposition 3.5 implies that M belongs to $\text{res } X$, and thus $M \in \mathcal{X}$.

Now, let $m \geq 1$. Let $\mathfrak{p} \in \min \text{NF}(M)$. Then we have $\dim \text{NF}(M_{\mathfrak{p}}) = 0$. Lemma 3.3(1) and the basis of the induction show that $M_{\mathfrak{p}} \in \text{add } \mathcal{X}_{\mathfrak{p}}$. Hence we can apply Lemma 3.3(2) to the nonempty finite set $\min \text{NF}(M)$, and obtain exact sequences

$$(3.6.1) \quad 0 \rightarrow K \rightarrow Y \rightarrow M \rightarrow 0,$$

$$(3.6.2) \quad 0 \rightarrow L \rightarrow M \oplus K \oplus R^{\oplus l} \rightarrow Y \rightarrow 0$$

with $Y \in \mathcal{X}$, $\text{NF}(L) \subseteq \text{NF}(M)$ and $\text{NF}(L) \cap \min \text{NF}(M) = \emptyset$. We easily see that $L \in \text{PD}(R)$ and that $\dim \text{NF}(L) < m$.

Let $\mathfrak{p} \in \text{NF}(L)$. Then \mathfrak{p} is in $\text{NF}(M)$, and hence $\text{pd } M_{\mathfrak{p}} \leq \text{pd } W_{\mathfrak{p}}$ for some $W \in \mathcal{X}$. There are exact sequences $0 \rightarrow K_{\mathfrak{p}} \rightarrow Y_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}} \rightarrow 0$ and $0 \rightarrow L_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}} \oplus K_{\mathfrak{p}} \oplus R_{\mathfrak{p}}^{\oplus l} \rightarrow Y_{\mathfrak{p}} \rightarrow 0$. We obtain inequalities

$$\begin{aligned} \text{pd } L_{\mathfrak{p}} &\leq \sup\{\text{pd}(M_{\mathfrak{p}} \oplus K_{\mathfrak{p}} \oplus R_{\mathfrak{p}}^{\oplus l}), \text{pd } Y_{\mathfrak{p}} - 1\} = \sup\{\text{pd } M_{\mathfrak{p}}, \text{pd } K_{\mathfrak{p}}, \text{pd } Y_{\mathfrak{p}} - 1\} \\ &\leq \sup\{\text{pd } M_{\mathfrak{p}}, \sup\{\text{pd } Y_{\mathfrak{p}}, \text{pd } M_{\mathfrak{p}} - 1\}, \text{pd } Y_{\mathfrak{p}} - 1\} \\ &\leq \sup\{\text{pd } M_{\mathfrak{p}}, \text{pd } Y_{\mathfrak{p}}\} \leq \sup\{\text{pd } W_{\mathfrak{p}}, \text{pd } Y_{\mathfrak{p}}\} = \text{pd } Z_{\mathfrak{p}}, \end{aligned}$$

where Z is either Y or W . Therefore, the module L satisfies the condition (2). Applying the induction hypothesis to L , we get $L \in \mathcal{X}$. By (3.6.2) we observe $M \in \mathcal{X}$. \blacksquare

Remark 3.7. It is known that the equality $\text{LPD}(R) = \text{PD}(R)$ always holds; see [7, 4.5].

Finally we give the proof of Theorem 1.4 from the Introduction:

Proof of Theorem 1.4. The implication (1) \Rightarrow (2) is a consequence of Theorem 3.6. The implication (3) \Rightarrow (1) follows by letting $X = R/\mathfrak{p}$ and localization at \mathfrak{p} .

Assume (2) and pick a prime ideal \mathfrak{p} which is not in $\text{Supp Tor}_i^R(N, X)$. Then $\text{Tor}_i^{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, X_{\mathfrak{p}}) = 0$, thus $\text{Tor}_j^{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, X_{\mathfrak{p}}) = 0$ for all $j \geq i$ by rigidity. Since $M \in \text{res } N$, it is easily seen that $\text{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, X_{\mathfrak{p}}) = 0$. \blacksquare

4. DOMINANT RESOLVING SUBCATEGORIES OVER A COHEN-MACAULAY RING

In this section, we consider dominant resolving subcategories over a Cohen-Macaulay ring. In the first three results below, we investigate, over a Cohen-Macaulay local ring, the structure of resolving closures containing syzygies of the residue field.

Lemma 4.1. *Let (R, \mathfrak{m}, k) be a Cohen-Macaulay local ring. Let \mathcal{X} be a resolving subcategory of $\text{mod } R$. Suppose that \mathcal{X} contains an R -module of depth 0. If $\Omega^n k$ is in \mathcal{X} for some $n \geq 0$, then k is in \mathcal{X} .*

Proof. Let X be an R -module in \mathcal{X} with $\text{depth } X = 0$. If $\text{NF}(X)$ is empty, then X is free. If $\text{NF}(X)$ is nonempty, then \mathfrak{m} belongs to it, and applying Proposition 3.2 to \mathfrak{m} , we find a module $X' \in \text{res } X$ with $\text{depth } X' = 0$ and $\text{NF}(X') = \{\mathfrak{m}\}$. Thus, we may assume that X is locally free on the punctured spectrum of R . There is an exact sequence $0 \rightarrow k \rightarrow X \rightarrow C \rightarrow 0$, and by Proposition 2.2(1) we get an exact sequence

$$0 \rightarrow \Omega C \rightarrow k \oplus R^{\oplus m} \rightarrow X \rightarrow 0.$$

Fix an integer $i \geq 1$. Applying Proposition 2.2(1) $((i-1)$ times) gives an exact sequence

$$0 \rightarrow \Omega^i C \rightarrow \Omega^{i-1} k \oplus R^{\oplus l} \rightarrow \Omega^{i-1} X \rightarrow 0$$

for some $l \geq 0$. Since C is locally free on the punctured spectrum of R , it belongs to $\text{res}_R k$ by [23, Theorem 2.4]. Hence $\Omega^i C$ is in $\text{res}_R(\Omega^i k)$. Therefore,

$$\Omega^i k \in \mathcal{X} \text{ implies } \Omega^{i-1} k \in \mathcal{X} \text{ for each } i \geq 1.$$

Thus the assertion follows. \blacksquare

Proposition 4.2. *Let (R, \mathfrak{m}, k) be a d -dimensional Cohen-Macaulay local ring. Let M be an R -module of depth t . Then $\Omega^t k$ belongs to $\text{res}(M \oplus \Omega^n k)$ for all $n \geq 0$.*

Proof. As $\text{res}_R(M \oplus \Omega_R^{i+1} k) \subseteq \text{res}_R(M \oplus \Omega_R^i k)$ for each $i \geq 0$, we may assume $n \geq d$. Then $\Omega_R^n k$ is a maximal Cohen-Macaulay R -module. Let $\mathbf{x} = x_1, \dots, x_t$ be a sequence of elements of R having the following properties:

- (1) The sequence \mathbf{x} is regular on both M and R ,
- (2) The element $\overline{x_i} \in \mathfrak{m}/(x_1, \dots, x_{i-1})$ is not in $(\mathfrak{m}/(x_1, \dots, x_{i-1}))^2$ for $1 \leq i \leq t$.

We can actually get such a sequence by choosing an element of \mathfrak{m} outside the ideal $\mathfrak{m}^2 + (x_1, \dots, x_{i-1})$ and the prime ideals in $\text{Ass}_R M/(x_1, \dots, x_{i-1})M \cup \text{Ass}_R R/(x_1, \dots, x_{i-1})$

for each $1 \leq i \leq t$. Putting $N = M \oplus \Omega_R^n k$, we see that \mathbf{x} is an N -sequence. Applying [20, Corollary 5.3] repeatedly, we have an isomorphism

$$\Omega_R^n k / \mathbf{x} \Omega_R^n k \cong \bigoplus_{j=0}^t (\Omega_{R/(\mathbf{x})}^{n-j} k)^{\oplus \binom{t}{j}}.$$

Hence $\text{res}_{R/(\mathbf{x})}(N/\mathbf{x}N) = \text{res}_{R/(\mathbf{x})}(M/\mathbf{x}M \oplus \Omega_R^n k / \mathbf{x} \Omega_R^n k)$ contains $\Omega_{R/(\mathbf{x})}^n k$. Since $\text{res}_{R/(\mathbf{x})}(N/\mathbf{x}N)$ contains the module $M/\mathbf{x}M$ of depth zero, it contains k by Lemma 4.1. It follows that $\Omega_R^t k$ belongs to $\text{res}_R(\Omega_R^t(N/\mathbf{x}N))$. The Koszul complex of \mathbf{x} with respect to N gives an exact sequence

$$0 \rightarrow N^{\oplus \binom{t}{t}} \rightarrow N^{\oplus \binom{t}{t-1}} \rightarrow \dots \rightarrow N^{\oplus \binom{t}{1}} \rightarrow N^{\oplus \binom{t}{0}} \rightarrow N/\mathbf{x}N \rightarrow 0,$$

which shows that $\Omega_R^t(N/\mathbf{x}N)$ is in $\text{res}_R N$ by [23, Lemma 4.3]. Therefore $\Omega_R^t k$ is in $\text{res}_R N$. \blacksquare

Corollary 4.3. *Let R be a Cohen-Macaulay local ring with residue field k . Let M be an R -module of depth t that is locally free on the punctured spectrum of R . Then $\text{res}(M \oplus \Omega^n k) = \text{res}(\Omega^t k \oplus \Omega^n k)$ for all $n \geq 0$.*

Proof. By [23, Theorem 2.4] the module M belongs to $\text{res } \Omega^t k$, which gives the inclusion $\text{res}(M \oplus \Omega^n k) \subseteq \text{res}(\Omega^t k \oplus \Omega^n k)$. The other inclusion follows from Proposition 4.2. \blacksquare

We slightly extends the definition of an dominant resolving subcategory as follows.

Definition 4.4. Let W be a subset of $\text{Spec } R$, and let \mathcal{X} be a resolving subcategory of $\text{mod } R$. We say that \mathcal{X} is *dominant* on W if for all $\mathfrak{p} \in W$ there exists $n \geq 0$ such that $\Omega^n \kappa(\mathfrak{p}) \in \text{add } \mathcal{X}_{\mathfrak{p}}$. (An dominant resolving subcategory of $\text{mod } R$ is nothing but a resolving subcategory that is dominant on $\text{Spec } R$.)

The main result of this section is the following theorem, which yields a criterion for a module to be in an dominant resolving subcategory.

Theorem 4.5. *Let R be a Cohen-Macaulay ring. Let \mathcal{X} be a resolving subcategory of $\text{mod } R$. Let M be an R -module such that \mathcal{X} is dominant on $\text{NF}(M)$. Then the following are equivalent:*

- (1) M belongs to \mathcal{X} ;
- (2) $\text{depth } M_{\mathfrak{p}} \geq \min_{X \in \mathcal{X}} \{\text{depth } X_{\mathfrak{p}}\}$ holds for all $\mathfrak{p} \in \text{NF}(M)$.

Proof. The implication (1) \Rightarrow (2) is trivial. To show the implication (2) \Rightarrow (1), we may assume that R is local with maximal ideal \mathfrak{m} by Lemma 3.3(1) and Proposition 3.4. Let M be an R -module with $\text{depth } M_{\mathfrak{p}} \geq \min_{X \in \mathcal{X}} \{\text{depth } X_{\mathfrak{p}}\}$ for all $\mathfrak{p} \in \text{Spec } R$. We induce on $m := \dim \text{NF}(M)$ to prove that M belongs to \mathcal{X} . When $m = -\infty$, the module M is projective, and is in \mathcal{X} . When $m = 0$, we have $\text{NF}(M) = \{\mathfrak{m}\}$. By assumption, $\Omega^n k$ is in \mathcal{X} for some $n \geq 0$. As M is locally free on the punctured spectrum, it follows from Corollary 4.3 that $\text{res}(M \oplus \Omega^n k) = \text{res}(\Omega^t k \oplus \Omega^n k)$, where $t = \text{depth } M$. We have $t = \text{depth } M \geq \min_{X \in \mathcal{X}} \{\text{depth } X\}$, which gives an R -module $X \in \mathcal{X}$ with $t \geq \text{depth } X =: s$. Hence $t - s \geq 0$, and $\Omega^t k = \Omega^{t-s}(\Omega^s k)$ belongs to $\text{res } \Omega^s k$. Proposition 4.2 implies that $\Omega^s k$ is in $\text{res}(X \oplus \Omega^n k)$, and we have

$$M \in \text{res}(M \oplus \Omega^n k) = \text{res}(\Omega^t k \oplus \Omega^n k) \subseteq \text{res}(\Omega^s k \oplus \Omega^n k) \subseteq \text{res}(X \oplus \Omega^n k) \subseteq \mathcal{X}.$$

Let us consider the case $m \geq 1$. Using Lemma 3.3(1) and the basis of the induction, we observe that $M_{\mathfrak{p}} \in \text{add } \mathcal{X}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \min \text{NF}(M)$. Lemma 3.3(2) implies that there exist exact sequences

$$(4.5.1) \quad 0 \rightarrow K \rightarrow X \rightarrow M \rightarrow 0,$$

$$(4.5.2) \quad 0 \rightarrow L \rightarrow M \oplus K \oplus R^{\oplus l} \rightarrow X \rightarrow 0$$

with $X \in \mathcal{X}$, $\text{NF}(L) \subseteq \text{NF}(M)$ and $\dim \text{NF}(L) < m$.

Here we claim that $\text{depth } L_{\mathfrak{p}} \geq \min_{Y \in \mathcal{X}} \{\text{depth } Y_{\mathfrak{p}}\}$ for all $\mathfrak{p} \in \text{NF}(L)$. Indeed, localizing the above exact sequences, we obtain inequalities

$$\begin{aligned} \text{depth } L_{\mathfrak{p}} &\geq \inf\{\text{depth}(M_{\mathfrak{p}} \oplus K_{\mathfrak{p}} \oplus R_{\mathfrak{p}}^{\oplus l}), \text{depth } X_{\mathfrak{p}} + 1\} \\ &= \inf\{\text{depth } M_{\mathfrak{p}}, \text{depth } K_{\mathfrak{p}}, \text{depth } X_{\mathfrak{p}} + 1\} \\ &\geq \inf\{\text{depth } M_{\mathfrak{p}}, \inf\{\text{depth } X_{\mathfrak{p}}, \text{depth } M_{\mathfrak{p}} + 1\}, \text{depth } X_{\mathfrak{p}} + 1\} \\ &\geq \inf\{\text{depth } M_{\mathfrak{p}}, \text{depth } X_{\mathfrak{p}}\} \geq \min_{Y \in \mathcal{X}} \{\text{depth } Y_{\mathfrak{p}}\}, \end{aligned}$$

where the last inequality follows from the fact that \mathfrak{p} is in $\text{NF}(M)$.

Now we can apply the induction hypothesis to L to get $L \in \mathcal{X}$. By (4.5.2), the module M belongs to \mathcal{X} . ■

The following result is a direct consequence of Theorem 4.5.

Corollary 4.6. *Let R be a Cohen-Macaulay ring with $\dim R = d < \infty$. Let \mathcal{X} be a resolving subcategory of $\text{mod } R$. The following are equivalent:*

- (1) \mathcal{X} is dominant;
- (2) For all $\mathfrak{p} \in \text{Spec } R$, there exists $n \geq 0$ such that $\Omega^n(R/\mathfrak{p}) \in \mathcal{X}$;
- (3) For all $\mathfrak{p} \in \text{Spec } R$, $\Omega^d(R/\mathfrak{p}) \in \mathcal{X}$.

Proof. (3) \Rightarrow (2) \Rightarrow (1): These implications are obvious.

(1) \Rightarrow (3): Let $\mathfrak{q} \in \text{Spec } R$. It is evident that $\Omega^d(R/\mathfrak{p})_{\mathfrak{q}}$ is a maximal Cohen-Macaulay $R_{\mathfrak{q}}$ -module. Hence the inequalities $\text{depth } \Omega^d(R/\mathfrak{p})_{\mathfrak{q}} \geq \dim R_{\mathfrak{q}} \geq \min_{X \in \mathcal{X}} \{\text{depth } X_{\mathfrak{q}}\}$ hold, and $\Omega^d(R/\mathfrak{p})$ belongs to \mathcal{X} by Theorem 4.5. ■

5. PROOFS OF THEOREMS 1.1 AND 1.2

This section is devoted to proving Theorem 1.1 and 1.2 by using the results obtained in the previous two sections. First of all, we define subcategories determined by local finiteness of homological dimensions.

Notation 5.1. We denote by \mathbf{h} either projective dimension pd or Cohen-Macaulay dimension CMdim . (For the definition of Cohen-Macaulay dimension, see [13, Definitions 3.2 and 3.2'].) Let $\mathbf{L}_{\mathbf{h}}(R)$ denote the subcategory of $\text{mod } R$ consisting of modules M with $\mathbf{h}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in \text{Spec } R$.

Here are some basic properties of projective and Cohen-Macaulay dimension and their consequences, which will be used without reference.

Lemma 5.2. *The following hold for $(\mathbf{h}, \mathbb{P}) = (\text{pd}, \text{regular}), (\text{CMdim}, \text{Cohen-Macaulay})$.*

- (1) Suppose that R is local.

- (a) It holds for an R -module M that $\mathbf{h}(M) \in \{-\infty\} \cup \mathbb{N} \cup \{\infty\}$, and that $\mathbf{h}(M) = -\infty \Leftrightarrow M = 0$.
- (b) If M, N are R -modules with $M \cong N$, then $\mathbf{h}(M) = \mathbf{h}(N)$.
- (c) For an R -module M , one has $\mathbf{CMdim} M \leq \mathbf{pd} M$, whose equality holds if $\mathbf{pd} M < \infty$.
- (d) If M is an R -module and N is a direct summand of M , then $\mathbf{h}(N) \leq \mathbf{h}(M)$.
- (e) If M is an R -module with $\mathbf{h}(M) < \infty$, then $\mathbf{h}(M) = \mathbf{depth} R - \mathbf{depth} M$.
- (f) For a nonzero R -module M , one has $\mathbf{h}(\Omega^n M) = \sup\{\mathbf{h}(M) - n, 0\}$.
- (g) If R satisfies \mathbb{P} , then $\mathbf{h}(M) < \infty$ for every R -module M .
- (2) For $M \in \mathbf{mod} R$ and $\mathfrak{p}, \mathfrak{q} \in \mathbf{Spec} R$ with $\mathfrak{p} \subseteq \mathfrak{q}$, one has $\mathbf{h}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \mathbf{h}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}})$.
- (3) If R satisfies \mathbb{P} , then $\mathbf{L}_h(R) = \mathbf{mod} R$.
- (4) One has $\mathbf{L}_{\mathbf{pd}}(R) = \mathbf{LPD}(R) = \mathbf{PD}(R)$.
- (5) For every R -module $M \in \mathbf{L}_h(R)$, the inequality $\mathbf{h}(M_{\mathfrak{p}}) \leq \mathbf{grade} \mathfrak{p}$ holds.

Proof. The statements (1)–(4) are well-known for \mathbf{pd} ; the second equality in (4) has already been observed in Remark 3.7. Those for \mathbf{CMdim} are stated in [13, §3]. Let us show the statement (5). There is an equality $\mathbf{grade} \mathfrak{p} = \inf\{\mathbf{depth} R_{\mathfrak{q}} \mid \mathfrak{q} \in V(\mathfrak{p})\}$, so we have $\mathbf{grade} \mathfrak{p} = \mathbf{depth} R_{\mathfrak{q}}$ for some $\mathfrak{q} \in V(\mathfrak{p})$. Since $\mathbf{h}(M_{\mathfrak{q}})$ is finite, it holds that $\mathbf{h}(M_{\mathfrak{p}}) \leq \mathbf{h}(M_{\mathfrak{q}}) = \mathbf{depth} R_{\mathfrak{q}} - \mathbf{depth} M_{\mathfrak{q}} \leq \mathbf{depth} R_{\mathfrak{q}} = \mathbf{grade} \mathfrak{p}$. \blacksquare

The following lemma will be necessary in both of the proofs of Theorems 1.1 and 1.2.

Lemma 5.3. *Let f be a grade consistent function on $\mathbf{Spec} R$. For each $\mathfrak{p} \in \mathbf{Spec} R$ there exists an R -module E satisfying:*

- (1) $\mathbf{pd} E < \infty$,
- (2) $\mathbf{pd} E_{\mathfrak{q}} \leq f(\mathfrak{q})$ for all $\mathfrak{q} \in \mathbf{Spec} R$,
- (3) $\mathbf{pd} E_{\mathfrak{p}} = f(\mathfrak{p})$.

Proof. Put $n = \mathbf{grade} \mathfrak{p}$. If $n = 0$, then $f(\mathfrak{p}) = 0$, and we can take $E := R$. Let $n \geq 1$. Choose an R -sequence $\mathbf{x} = x_1, \dots, x_n$ in \mathfrak{p} . It is easy to observe that \mathfrak{p} belongs to $\mathbf{NF}(R/(\mathbf{x}))$. Proposition 3.2 implies that there exists an R -module $X \in \mathbf{res}(R/(\mathbf{x}))$ with $\mathbf{NF}(X) = V(\mathfrak{p})$ and $\mathbf{depth} X_{\mathfrak{q}} = \inf\{\mathbf{depth} R_{\mathfrak{q}}/\mathbf{x}R_{\mathfrak{q}}, \mathbf{depth} R_{\mathfrak{q}}\} = \mathbf{depth} R_{\mathfrak{q}} - n$ for all $\mathfrak{q} \in V(\mathfrak{p})$. Note that $n - f(\mathfrak{p}) \geq 0$. Put $E := \Omega^{n-f(\mathfrak{p})} X$.

- (1) As the R -module $R/(\mathbf{x})$ has finite projective dimension, so does X , and so does E .
- (2) If \mathfrak{q} is not in $V(\mathfrak{p})$, then $\mathfrak{q} \notin \mathbf{NF}(X)$. Hence $X_{\mathfrak{q}}$ is $R_{\mathfrak{q}}$ -free, and so is $E_{\mathfrak{q}}$. Thus we may assume $\mathfrak{q} \in V(\mathfrak{p})$. Then we have

$$\begin{aligned} \mathbf{pd} E_{\mathfrak{q}} &= \mathbf{pd} \Omega^{n-f(\mathfrak{p})} X_{\mathfrak{q}} = \sup\{\mathbf{pd} X_{\mathfrak{q}} - n + f(\mathfrak{p}), 0\} \\ &= \sup\{\mathbf{depth} R_{\mathfrak{q}} - \mathbf{depth} X_{\mathfrak{q}} - n + f(\mathfrak{p}), 0\} = \sup\{f(\mathfrak{p}), 0\} = f(\mathfrak{p}) \leq f(\mathfrak{q}). \end{aligned}$$

- (3) The above equalities imply that $\mathbf{pd} E_{\mathfrak{p}} = f(\mathfrak{p})$. \blacksquare

Now we are on the stage to achieve the main purpose of this section.

Proof of Theorem 1.1. It is easy to verify that ϕ, ψ are well-defined maps.

Let \mathcal{X} be a resolving subcategory contained in $\mathbf{PD}(R)$. Then

$$\psi\phi(\mathcal{X}) = \{M \in \mathbf{mod} R \mid \mathbf{pd} M_{\mathfrak{p}} \leq \max_{X \in \mathcal{X}} \{\mathbf{pd} X_{\mathfrak{p}}\} \text{ for all } \mathfrak{p} \in \mathbf{Spec} R\} \supseteq \mathcal{X}.$$

It follows from Theorem 3.6 that $\psi\phi(\mathcal{X}) = \mathcal{X}$.

Let f be a grade consistent function on $\text{Spec } R$. Fix a prime ideal \mathfrak{p} . We have

$$\phi\psi(f)(\mathfrak{p}) = \max\{\text{pd } X_{\mathfrak{p}} \mid X \in \text{mod } R \text{ with } \text{pd } X_{\mathfrak{q}} \leq f(\mathfrak{q}) \text{ for all } \mathfrak{q} \in \text{Spec } R\} \leq f(\mathfrak{p}).$$

It follows from Lemma 5.3 that $\phi\psi(f)(\mathfrak{p}) = f(\mathfrak{p})$. Therefore $\phi\psi(f) = f$. \blacksquare

Proof of Theorem 1.2. Fix $\mathfrak{p} \in \text{Spec } R$. We see from [4, Theorem 1.1] that the number $n := \text{Rfd}_R(R/\mathfrak{p})$ is finite. It follows from [8, Proposition (2.3) and Theorem (2.8)] that $\text{CMdim}_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) - n = \text{Rfd}_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) - n \leq 0$, and hence $\text{CMdim}_{R_{\mathfrak{q}}} \Omega^n(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) = \sup\{\text{CMdim}_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) - n, 0\} = 0 \leq f(\mathfrak{q})$ for each prime ideal \mathfrak{q} of R . Therefore $\Omega^n(R/\mathfrak{p})$ is in $\psi(f)$, which implies that $\psi(f)$ is dominant. Now we easily check that ϕ, ψ are well-defined.

Let \mathcal{X} be an dominant resolving subcategory of $\text{mod } R$. Then we have

$$\psi\phi(\mathcal{X}) = \{M \in \text{mod } R \mid \text{depth } M_{\mathfrak{p}} \geq \min_{X \in \mathcal{X}} \{\text{depth } X_{\mathfrak{p}}\} \text{ for all } \mathfrak{p} \in \text{Spec } R\} \supseteq \mathcal{X}.$$

By virtue of Theorem 4.5, the equality $\psi\phi(\mathcal{X}) = \mathcal{X}$ holds.

Let f be a grade consistent function on $\text{Spec } R$, and let $\mathfrak{p} \in \text{Spec } R$. Then

$$\begin{aligned} \phi\psi(f)(\mathfrak{p}) &= \max\{\text{CMdim } X_{\mathfrak{p}} \mid X \in \text{mod } R \text{ with } \text{CMdim } X_{\mathfrak{q}} \leq f(\mathfrak{q}) \text{ for all } \mathfrak{q} \in \text{Spec } R\} \\ &\leq f(\mathfrak{p}). \end{aligned}$$

Taking an R -module E as in Lemma 5.3, we deduce that $\phi\psi(f)(\mathfrak{p}) = f(\mathfrak{p})$. It follows that $\phi\psi(f) = f$. \blacksquare

As a common consequence of Theorems 1.1 and 1.2, we immediately obtain a complete classification of the resolving subcategories over a regular ring:

Corollary 5.4. *Let R be a regular ring. Then there exist mutually inverse bijections*

$$\left\{ \begin{array}{c} \text{Resolving subcategories} \\ \text{of } \text{mod } R \end{array} \right\} \begin{array}{c} \xrightarrow{\phi} \\ \xleftarrow{\psi} \end{array} \left\{ \begin{array}{c} \text{Grade consistent functions} \\ \text{on } \text{Spec } R \end{array} \right\}.$$

Here ϕ, ψ are defined as follows:

$$\begin{aligned} \phi(\mathcal{X}) &= [\mathfrak{p} \mapsto \max_{X \in \mathcal{X}} \{\text{pd } X_{\mathfrak{p}}\}], \\ \psi(f) &= \{M \in \text{mod } R \mid \text{pd } M_{\mathfrak{p}} \leq f(\mathfrak{p}) \text{ for all } \mathfrak{p} \in \text{Spec } R\}. \end{aligned}$$

Recall that R has (at most) an *isolated singularity* if $R_{\mathfrak{p}}$ is a regular local ring for each nonmaximal prime ideal \mathfrak{p} of R . As an application of Theorem 1.2, we have the following.

Corollary 5.5. *Let (R, \mathfrak{m}, k) be a d -dimensional Cohen-Macaulay local ring with an isolated singularity. Then one has the following one-to-one correspondences.*

$$\left\{ \begin{array}{c} \text{Resolving subcategories of } \text{mod } R \\ \text{containing } \Omega^d k \end{array} \right\} \begin{array}{c} \xrightarrow{\phi} \\ \xleftarrow{\psi} \end{array} \left\{ \begin{array}{c} \text{Grade consistent functions} \\ \text{on } \text{Spec } R \end{array} \right\},$$

where ϕ, ψ are defined by:

$$\begin{aligned} \phi(\mathcal{X}) &= [\mathfrak{p} \mapsto \text{ht } \mathfrak{p} - \min_{X \in \mathcal{X}} \{\text{depth } X_{\mathfrak{p}}\}], \\ \psi(f) &= \{M \in \text{mod } R \mid \text{depth } M_{\mathfrak{p}} \geq \text{ht } \mathfrak{p} - f(\mathfrak{p}) \text{ for all } \mathfrak{p} \in \text{Spec } R\}. \end{aligned}$$

Proof. According to Theorem 1.2, it is enough to check that a resolving subcategory \mathcal{X} of $\mathbf{mod} R$ is dominant if and only if $\Omega^d k \in \mathcal{X}$. The ‘only if’ part is obvious. As to the ‘if’ part, take a prime ideal \mathfrak{p} of R . Let us show $\Omega^d \kappa(\mathfrak{p}) \in \mathbf{add} \mathcal{X}_{\mathfrak{p}}$. It is clear if $\mathfrak{p} = \mathfrak{m}$, so assume $\mathfrak{p} \neq \mathfrak{m}$. Then $R_{\mathfrak{p}}$ is regular, and $\Omega^d \kappa(\mathfrak{p})$ is a free $R_{\mathfrak{p}}$ -module. Hence it is in $\mathbf{add} \mathcal{X}_{\mathfrak{p}}$. \blacksquare

6. CLASSIFICATION FOR LOCALLY HYPERSURFACE RINGS

In this section, as an application of our results obtained so far, we give classification of resolving subcategories over a locally hypersurface ring. Let us start by giving a criterion for a module to be in a resolving subcategory:

Theorem 6.1. *Let R be a locally hypersurface ring. Let \mathcal{X} be a resolving subcategory of $\mathbf{mod} R$. Then an R -module M belongs to \mathcal{X} if and only if it satisfies the following conditions:*

- (1) $\mathrm{IPD}(M) \subseteq \mathrm{IPD}(\mathcal{X})$,
- (2) $\mathrm{depth} M_{\mathfrak{p}} \geq \min_{X \in \mathcal{X}} \{\mathrm{depth} X_{\mathfrak{p}}\}$ for all $\mathfrak{p} \in \mathrm{NF}(M)$.

Proof. The ‘only if’ part is trivial. In the following we prove the ‘if’ part. According to Lemma 3.3 and Proposition 3.4, we may assume that R is a local hypersurface. Then M admits a so-called finite projective hull, namely, there exists an exact sequence

$$0 \rightarrow M \rightarrow P \rightarrow C \rightarrow 0$$

of R -modules such that P has finite projective dimension and that C is maximal Cohen-Macaulay (cf. [3, Theorem 1.8]). Note that the inequality $\mathrm{depth} M_{\mathfrak{p}} \geq \min_{X \in \mathcal{X}} \{\mathrm{depth} X_{\mathfrak{p}}\}$ holds for all $\mathfrak{p} \in \mathrm{Spec} R$, as $R \in \mathcal{X}$.

(a) Let $\mathfrak{p} \in \mathrm{Spec} R$. We have

$$\mathrm{depth} P_{\mathfrak{p}} \geq \inf\{\mathrm{depth} M_{\mathfrak{p}}, \mathrm{depth} C_{\mathfrak{p}}\} = \mathrm{depth} M_{\mathfrak{p}} \geq \min_{X \in \mathcal{X}} \{\mathrm{depth} X_{\mathfrak{p}}\},$$

and hence $\mathrm{depth} P_{\mathfrak{p}} \geq \mathrm{depth} X_{\mathfrak{p}}$ for some $X \in \mathcal{X}$. The proof of [9, Lemma 4.13] yields a module $Y \in \mathcal{X}$ with $\mathrm{pd} Y < \infty$ and $\mathrm{depth} Y_{\mathfrak{p}} \leq \mathrm{depth} X_{\mathfrak{p}}$. Thus $Y \in \mathcal{X} \cap \mathrm{PD}(R)$ and $\mathrm{pd} P_{\mathfrak{p}} \leq \mathrm{pd} Y_{\mathfrak{p}}$. Applying Theorem 3.6 to the resolving subcategory $\mathcal{X} \cap \mathrm{PD}(R)$, we see that $P \in \mathcal{X}$.

(b) It holds that

$$\mathrm{NF}(C) = \mathrm{IPD}(C) = \mathrm{IPD}(M) \subseteq \mathrm{IPD}(\mathcal{X}) = \mathrm{NF}(\mathcal{X} \cap \mathrm{CM}(R)),$$

where the inclusion relation $\mathrm{IPD}(\mathcal{X}) \subseteq \mathrm{NF}(\mathcal{X} \cap \mathrm{CM}(R))$ is shown by taking a high enough syzygy. Note that $\mathcal{X} \cap \mathrm{CM}(R)$ is a resolving subcategory of $\mathbf{mod} R$. Since R is a local hypersurface, we observe by [23, Main Theorem] that C is in $\mathcal{X} \cap \mathrm{CM}(R)$, and hence in \mathcal{X} .

Now, it follows from (a) and (b) that M belongs to \mathcal{X} . \blacksquare

Making use of Theorem 6.1, we can prove Theorem 1.3.

Proof of Theorem 1.3. It is easy to check that Φ, Ψ are well-defined. Theorem 6.1 yields $\Psi\Phi = 1$. Fix a specialization closed subset W of $\mathrm{Sing} R$ and a grade consistent

function f on $\text{Spec } R$. Using Lemma 5.3, we observe that $\phi(\Psi(W, f)) = f$ holds. It now remains only to prove the equality $\text{IPD}(\Psi(W, f)) = W$.

It is easily seen that $\text{IPD}(\Psi(W, f))$ is contained in W . Let $\mathfrak{p} \in W$. Put $n = \text{Rfd}_R(R/\mathfrak{p})$ (cf. [4, Theorem 1.1]) and $M = \Omega_R^n(R/\mathfrak{p})$. Since $\text{IPD}(M) = \text{IPD}(R/\mathfrak{p}) = V(\mathfrak{p}) \subseteq W$, the module M is in $\text{IPD}^{-1}(W)$. Take any prime ideal \mathfrak{q} of R . We see that $\text{CMdim}_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) = \text{Rfd}_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) \leq n$ by [8, Proposition (2.3) and Theorem (2.8)], and hence the $R_{\mathfrak{q}}$ -module $M_{\mathfrak{q}}$ is maximal Cohen-Macaulay. Therefore $\text{depth } M_{\mathfrak{q}} \geq \text{ht } \mathfrak{q} \geq \text{ht } \mathfrak{q} - f(\mathfrak{q})$ holds for every $\mathfrak{q} \in \text{Spec } R$, which shows that M belongs to $\Psi(W, f)$. Since $\mathfrak{p} \in V(\mathfrak{p}) = \text{IPD}(M)$, we conclude that $\text{IPD}(\Psi(W, f)) = W$. ■

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